ON THE ℓ-MODULAR COMPOSITION FACTORS OF THE STEINBERG REPRESENTATION

MEINOLF GECK

ABSTRACT. Let G be a finite group of Lie type and St_k be the Steinberg representation of G, defined over a field k. We are interested in the case where k has prime characteristic ℓ and St_k is reducible. Tinberg has shown that the socle of St_k is always simple. We give a new proof of this result in terms of the Hecke algebra of G with respect to a Borel subgroup and show how to identify the simple socle of St_k among the principal series representations of G. Furthermore, we determine the composition length of St_k when $G = \mathrm{GL}_n(q)$ or G is a finite classical group and ℓ is a so-called linear prime.

1. INTRODUCTION

Let G be a finite group of Lie type and St_k be the Steinberg representation of G, defined over a field k. Steinberg [30] showed that St_k is irreducible if and only if $[G : B]1_k \neq 0$ where B is a Borel subgroup of G. We shall be concerned here with the case where St_k is reducible. There is only very little general knowledge about the structure of St_k in this case. We mention the works of Tinberg [33] (on the socle of St_k), Hiss [18] and Khammash [26] (on trivial composition factors of St_k) and Gow [14] (on the Jantzen filtration of St_k).

One of the most important open questions in this respect seems to be to find a suitable bound on the length of a composition series of St_k. Typically, this problem is related to quite subtle information about decomposition numbers; see, for example, Landrock–Michler [27] and Okuyama–Waki [29] where this is solved for groups with a BN-pair of rank 1. For groups of larger BN-rank, this problem is completely open.

In this paper, we discuss two aspects of this problem.

Firstly, Tinberg [33] has shown that the socle of St_k is always simple, using results of Green [15] applied to the endomorphism algebra of the permutation module $k[G/U]$ where U is a maximal unipotent subgroup. After some preparations in Sections 2, we show in Section 3 that a similar argument works with U replaced by B. Since the corresponding endomorphism algebra (or "Hecke algebra") is much easier to describe and its representation theory is quite well understood, this provides new additional information. For example, if $G = \mathrm{GL}_n(q)$, then we can identify the partition of n which labels the socle of St_k in James' [23] parametrisation of the unipotent simple modules of G; see Example 3.6. Quite remarkably, this involves a particular case of the "Mullineux involution" — and an analogue of this involution for other types of groups.

2000 *Mathematics Subject Classification.* Primary 20C33, Secondary 20C20.

In another direction, we consider the partition of the simple kG-modules into Harish-Chandra series, as defined by Hiss [19]. Dipper and Gruber [6] have developed a quite general framework for this purpose, in terms of so-called "projective restriction systems". In Section 4, we shall present a simplified, self-contained version of parts of this framework which is tailored towards applications to St_k. This yields, first of all, new proofs of some of the results of Szechtman [32] on St_k for $G = \mathrm{GL}_n(q)$; moreover, in Example 4.9, we obtain an explicit formula for the composition length of St_k in this case. Analogous results are derived for groups of classical type in the so-called "linear prime" case, based on [9], [16], [17]. For example, St_k is seen to be multiplicity-free with a unique simple quotient in these cases — a property which does not hold in general for non-linear primes.

2. The Steinberg module and the Hecke algebra

Let G be a finite group and $B, N \subseteq G$ be subgroups which satisfy the axioms for an "algebraic group with a split BN-pair" in [2, §2.5]. We just recall explicitly those properties of G which will be important for us in the sequel. Firstly, there is a prime number p such that we have a semidirect product decomposition $B = U \rtimes H$ where $H = B \cap N$ is an abelian group of order prime to p and U is a normal p-subgroup of B. The group H is normal in N and $W = N/H$ is a finite Coxeter group with a canonically defined generating set S; let $l \colon W \to \mathbb{N}_0$ be the corresponding length function. For each $w \in W$, let $n_w \in N$ be such that $Hn_w = w$. Then we have the Bruhat decomposition

$$G = \coprod_{w \in W} Bn_wB = \coprod_{w \in W} Bn_wU,$$

where the second equality holds since $B = U \rtimes H$ and H is normal in N.

Next, there is a refinement of the above decomposition. Let $w_0 \in W$ be the unique element of maximal length; we have $w_0^2 = 1$. Let $n_0 \in N$ be a representative of w_0 and $V := n_0^{-1}Un_0$; then $U \cap V = H$. For $w \in W$, let $U_w := U \cap n_w^{-1}Vn_w$. (Note that V, U_w do not depend on the choice of n_0, n_w since U is normalised by H.) Then we have the following sharp form of the Bruhat decomposition:

$$G = \coprod_{w \in W} Bn_wU_w, \qquad \text{with uniqueness of expressions,}$$

that is, every $g \in Bn_wB$ can be uniquely written as $g = bn_wu$ where $b \in B$ and $u \in U_w$. It will occasionally be useful to have a version of this where the order of factors is reversed: By inverting elements, we obtain

$$G = \coprod_{w \in W} U_{w^{-1}}n_wB, \qquad \text{with uniqueness of expressions.}$$

Now let R be a commutative ring (with identity 1_R) and RG be the group algebra of G over R. All our RG-modules will be left modules and, usually, finitely generated. For any subgroup $X \subseteq G$, we denote by R_X the trivial RX-module. Let $\underline{\mathfrak{b}} := \sum_{b \in B} b \in RG$. Then $RG\underline{\mathfrak{b}}$ is an RG-module which is canonically isomorphic to the induced module $\mathrm{Ind}_B^G(R_B)$.

In fact, this realization of $\mathrm{Ind}_B^G(R_B)$ will be particularly suited for our purposes, as we shall see below when we consider its endomorphism algebra.

Theorem 2.1 (Steinberg [30]). *Consider the RG-submodule*

$$\mathrm{St}_R := RG\mathfrak{e} \subseteq RG\underline{\mathfrak{b}} \qquad \textit{where} \qquad \mathfrak{e} := \sum_{w\in W}(-1)^{l(w)} n_w\underline{\mathfrak{b}}.$$

(i) *The set $\{u\mathfrak{e} \mid u \in U\}$ is an R-basis of St_k. Thus, St_R is free over R of rank $|U|$.*
(ii) *Assume that R is a field. Then St_R is an (absolutely) irreducible RG-module if and only if $[G : B]1_R \neq 0$.*

(*Note about the proof:* Steinberg uses a list of 14 axioms concerning finite Chevalley groups and their twisted versions; all these axioms are known to hold in the abstract setting of "algebraic groups with a split BN-pair"; see [2, §2.5 and Prop. 2.6.1].)

When $R = k$ is a field, Tinberg [33, Theorem 4.10] determined the socle of St_k and showed that this is simple. An essential ingredient in Tinberg's proof are Green's results [15] on the Hom functor, applied to the endomorphism algebra of the kG-module $kG\underline{\mathfrak{u}}_1$, where $\underline{\mathfrak{u}}_1 = \sum_{u\in U} u$. There is a description of this algebra in terms of generators and relations, and this is used in order to study the indecomposable direct summands of $kG\underline{\mathfrak{u}}_1$. Our aim is to show that an analogous argument works directly with the module $kG\underline{\mathfrak{b}}$, whose endomorphism algebra has a much simpler description.

So let again R be any commutative ring (with 1_R), and consider the Hecke algebra

$$\mathscr{H}_R = \mathscr{H}_R(G, B) := \mathrm{End}_{RG}(RG\underline{\mathfrak{b}})^{\mathrm{opp}}.$$

Following Green [15], a connection between (left) RG-modules and (left) $\mathscr{H}_R$-modules is established through the Hom functor

$$\mathfrak{F}_R\colon RG\text{-modules} \to \mathscr{H}_R\text{-modules}, \qquad M \mapsto \mathfrak{F}_R(M) := \mathrm{Hom}_{RG}(RG\underline{\mathfrak{b}}, M),$$

where $\mathfrak{F}_R(M)$ is a left $\mathscr{H}_R$-module via $\mathscr{H}_R \times \mathfrak{F}_R(M) \to \mathfrak{F}_R(M)$, $(h, \alpha) \mapsto \alpha \circ h$. (See also [7, §2.C] where this Hom functor is studied in a somewhat more general context.) Note that, by [15, (1.3)], we have an isomorphism of R-modules

$$\mathrm{Fix}_B(M) := \{m \in M \mid b.m = m \text{ for all } b \in B\} \xrightarrow{\sim} \mathfrak{F}_R(M),$$

which takes $m \in \mathrm{Fix}_B(M)$ to the map $\theta_m\colon RG\underline{\mathfrak{b}} \to M$, $g\underline{\mathfrak{b}} \mapsto gm$ ($g \in G$).

Now, $\mathscr{H}_R$ is free over R with a standard basis $\{T_w \mid w \in W\}$, where the endomorphism $T_w\colon RG\underline{\mathfrak{b}} \to RG\underline{\mathfrak{b}}$ is given by

$$T_w(g\underline{\mathfrak{b}}) = \sum_{g'B\in G/B \text{ with } g^{-1}g'\in Bn_wB} g'\underline{\mathfrak{b}} \qquad (g \in G).$$

The multiplication is given as follows. Let $w \in W$, $s \in S$ and write $q_s := |U_s|1_R$. Then

$$T_sT_w = \begin{cases} T_{sw} & \text{if } l(sw) > l(w), \\ q_sT_{sw} + (q_s - 1)T_w & \text{if } l(sw) < l(w). \end{cases}$$

(See [13, §8.4] for a proof and further details.) The crucial step in our discussion consists of determining the $\mathscr{H}_R$-module $\mathfrak{F}_R(\mathrm{St}_R)$. This will rely on the following basic identity, an analogous version of which was shown by Tinberg [33, 4.9] for the action of the standard basis elements of the endomorphism algebra of $kG\underline{\mathfrak{u}}_1$ (where k is a field).

Lemma 2.2. *We have $T_w(\mathfrak{e}) = (-1)^{l(w)}\mathfrak{e}$ for all $w \in W$.*

Proof. It is sufficient to show that $T_s(\mathfrak{e}) = -\mathfrak{e}$ for $s \in S$. Now, by definition, we have

$$T_s(\mathfrak{e}) = \sum_{w\in W} (-1)^{l(w)} T_s(n_w\underline{\mathfrak{b}}) = \sum_{w\in W} (-1)^{l(w)} \sum_{gB} g\underline{\mathfrak{b}}$$

where the second sum runs over all cosets $gB \in G/B$ such that $n_w^{-1}g \in Bn_sB$. By the sharp form of the Bruhat decomposition, a set of representatives for these cosets is given by $\{n_{ws}\} \cup \{n_wvn_s \mid 1 \neq v \in U_s\}$. This yields

$$T_s(\mathfrak{e}) = \sum_{w\in W} (-1)^{l(w)} n_{ws}\underline{\mathfrak{b}} + \sum_{w\in W}\sum_{1\neq v\in U_s} (-1)^{l(w)} n_wvn_s\underline{\mathfrak{b}}.$$

Since $l(ws) = l(w) + 1$ for $w \in W$, the first sum equals $-\mathfrak{e}$. So it suffices to show that

$$\sum_{w\in W} (-1)^{l(w)}\kappa_w = 0 \qquad \text{where} \qquad \kappa_w := \sum_{1\neq v\in U_s} n_wvn_s\underline{\mathfrak{b}}.$$

Let $1 \neq v \in U_s$. Since $P_s = B \cup Bn_sB$ is a parabolic subgroup of G, we have $n_s^{-1}vn_s \in P_s$. By the sharp form of the Bruhat decomposition, $n_s^{-1}vn_s \notin B$ and so $n_s^{-1}vn_s = v'n_sb_v$ where $v' \in U_s$ and $b_v \in B$ are uniquely determined by v. Hence, we have $n_wvn_s\underline{\mathfrak{b}} = n_wn_sv'n_sb_v\underline{\mathfrak{b}} = n_{ws}v'n_s\underline{\mathfrak{b}}$ and so

$$\kappa_w = \sum_{1\neq v\in U_s} n_wvn_s\underline{\mathfrak{b}} = \sum_{1\neq v\in U_s} n_{ws}v'n_s\underline{\mathfrak{b}} = \sum_{1\neq v\in U_s} n_{ws}vn_s\underline{\mathfrak{b}} = \kappa_{ws},$$

where the third equality holds since, by [33, 2.1], the map $v \mapsto v'$ is a permutation of $U_s \setminus \{1\}$. Consequently, we have

$$\sum_{w\in W} (-1)^{l(w)}\kappa_w = \sum_{w\in W} (-1)^{l(w)}\kappa_{ws} = \sum_{w\in W} (-1)^{l(ws)}\kappa_w = -\sum_{w\in W} (-1)^{l(w)}\kappa_w.$$

We conclude that the identity $\sum_{w\in W}(-1)^{l(w)}\kappa_w = 0$ holds if $R = \mathbb{Z}$. For R arbitrary, we apply the canonical map $\mathbb{Z}G \to RG$ and conclude that this identity remains valid in RG. (Such an argument was already used by Steinberg in the proof of [30, Lemma 2].) $\square$

Corollary 2.3. *We have $\mathfrak{F}_R(\mathrm{St}_R) = \langle\theta_{\underline{\mathfrak{u}}_1\mathfrak{e}}\rangle_R$ and the action of $\mathscr{H}_R$ on this R-module of rank 1 is given by the algebra homomorphism $\varepsilon\colon \mathscr{H}_R \to R$, $T_w \mapsto (-1)^{l(w)}$.*

Proof. Since $\{u\mathfrak{e} \mid u \in U\}$ is an R-basis of St_R and H normalises U, we have $\mathrm{Fix}_B(\mathrm{St}_R) = \langle\underline{\mathfrak{u}}_1\mathfrak{e}\rangle_R$ and so $\mathfrak{F}_R(\mathrm{St}_R) = \langle\theta_{\underline{\mathfrak{u}}_1\mathfrak{e}}\rangle_R$. It remains to show that $T_s.\theta_{\underline{\mathfrak{u}}_1\mathfrak{e}} = -\theta_{\underline{\mathfrak{u}}_1\mathfrak{e}}$ for all $s \in S$.

Since $\mathfrak{F}_R(\mathrm{St}_R)$ has rank 1, we have $T_s.\theta_{\underline{\mathfrak{u}}_1\mathfrak{e}} = \lambda\theta_{\underline{\mathfrak{u}}_1\mathfrak{e}}$ for some $\lambda \in R$. This implies that

$$\lambda\underline{\mathfrak{u}}_1\mathfrak{e} = \lambda\theta_{\underline{\mathfrak{u}}_1\mathfrak{e}}(\underline{\mathfrak{b}}) = (T_s.\theta_{\underline{\mathfrak{u}}_1\mathfrak{e}})(\underline{\mathfrak{b}}) = (\theta_{\underline{\mathfrak{u}}_1\mathfrak{e}} \circ T_s)(\underline{\mathfrak{b}}) = \sum_{gB\in G/B \text{ with } g\in Bn_sB} g\underline{\mathfrak{u}}_1\mathfrak{e}.$$

Thus, the assertion that $\lambda = -1$ is equivalent to the following identity:

$$(*) \qquad \sum_{gB\in G/B \text{ with } g\in Bn_sB} g\underline{\mathfrak{u}}_1\mathfrak{e} = -\underline{\mathfrak{u}}_1\mathfrak{e}.$$

One can either work this out directly by an explicit computation (using the various "structural equations" in [30], [33]), or one can argue as follows. Lemma 2.2 shows that

$$\lambda\theta_{\underline{\mathfrak{u}}_1\mathfrak{e}}(\mathfrak{e}) = (T_s.\theta_{\underline{\mathfrak{u}}_1\mathfrak{e}})(\mathfrak{e}) = (\theta_{\underline{\mathfrak{u}}_1\mathfrak{e}} \circ T_s)(\mathfrak{e}) = -\theta_{\underline{\mathfrak{u}}_1\mathfrak{e}}(\mathfrak{e}).$$

Furthermore, by Steinberg [30, Lemma 2], we have

$$\theta_{\underline{\mathfrak{u}}_1\mathfrak{e}}(\mathfrak{e}) = \sum_{w\in W}(-1)^{l(w)}n_w\underline{\mathfrak{u}}_1\mathfrak{e} = \sum_{w\in W}\sum_{u\in U}(-1)^{l(w)}n_wu\mathfrak{e} = [G:B]\mathfrak{e}.$$

Thus, if $R = \mathbb{Z}$, then $\theta_{\underline{\mathfrak{u}}_1 e}(\mathfrak{e}) \neq 0$; consequently, in this case, we do have $\lambda = -1$ and so $(*)$ holds for $R = \mathbb{Z}$. As in the above proof, it follows that $(*)$ holds for any R. $\square$

Remark 2.4. Assume that R is an integral domain and that we have a decomposition $RG\underline{\mathfrak{b}} = M_1 \oplus \cdots \oplus M_r$ where each M_j is an indecomposable RG-module. Since $\{T_w \mid w \in W\}$ is an R-basis of $\mathscr{H}_R$, Lemma 2.2 implies that every idempotent in $\mathscr{H}_R$ either acts as the identity on St_R or as 0. It easily follows that there is a unique i such that $\mathrm{St}_R \subseteq M_i$. In analogy to Tinberg [33, 4.10], we call this M_i the *Steinberg component* of $RG\underline{\mathfrak{b}}$.

As observed by Khammash [25, (3.10)], the above argument actually shows that

$$\mathrm{St}_R \subseteq \{m \in RG\underline{\mathfrak{b}} \mid T_w(m) = (-1)^{l(w)}m \text{ for all } w \in W\} \subseteq M_i.$$

Then Khammash [26, Cor. §3] proved that the first inequality always is an equality.

Remark 2.5. At some places in the discussion below, it will be convenient or even necessary to assume that G is a true finite group of Lie type, as in [2, §1.18]. Thus, using the notation in [*loc. cit.*], we have $G = \mathbf{G}^F$ where $\mathbf{G}$ is a connected reductive algebraic group $\mathbf{G}$ over $\overline{\mathbb{F}}_p$ and $F\colon \mathbf{G} \to \mathbf{G}$ is an endomorphism such that some power of F is a Frobenius map. Then the ingredients of the BN-pair in G will also be derived from $\mathbf{G}$: we have $B = \mathbf{B}^F$ where $\mathbf{B}$ is an F-stable Borel subgroup of $\mathbf{G}$ and $H = \mathbf{T}_0^F$ where $\mathbf{T}_0$ is an F-stable maximal torus contained in $\mathbf{B}$; furthermore, $N = N_{\mathbf{G}}(\mathbf{T}_0)^F$ and $U = \mathbf{U}^F$ where $\mathbf{U}$ is the unipotent radical of $\mathbf{B}$. This set-up leads to the following two definitions.

(1) We define a real number $q > 0$ by the condition that $|U| = q^{|\Phi|/2}$ where Φ is the root system of $\mathbf{G}$ with respect to $\mathbf{T}_0$. Then there are positive integers $c_s > 0$ such that $|U_s| = q^{c_s}$ for all $s \in S$; see [2, §2.9]. Consequently, the relations in $\mathscr{H}_R$ read:

$$T_sT_w = \begin{cases} T_{sw} & \text{if } l(sw) > l(w), \\ q^{c_s}T_{sw} + (q^{c_s}-1)T_w & \text{if } l(sw) < l(w). \end{cases}$$

(2) The commutator subgroup $[\mathbf{U}, \mathbf{U}]$ is an F-stable closed connected normal subgroup of $\mathbf{U}$. We define the subgroup $U^* := [\mathbf{U}, \mathbf{U}]^F \subseteq U$. Then $[U, U] \subseteq U^*$. (In most cases, we have $U^* = [U, U]$ but there are exceptions when q is very small; see the remarks in [31, p. 258].) The definition of U^* will be needed in Section 4, where we shall consider group homomorphisms $\sigma \colon U \to R^\times$ such that $U^* \subseteq \ker(\sigma)$.

3. The socle of the Steinberg module

We keep the general setting of the previous section and assume now that $R = k$ is a field and $\ell := \mathrm{char}(k) \neq p$; thus, the parameters of the endomorphism algebra $\mathscr{H}_k$ satisfy $q_s \neq 0$ for all $s \in S$. With this assumption, we have the following two results:

- (A) Every simple submodule of $kG\underline{\mathfrak{b}}$ is isomorphic to a factor module of $kG\underline{\mathfrak{b}}$, and vice versa; see Hiss [19, Theorem 5.8] where this is proved much more generally.
- (B) $\mathscr{H}_k$ is a quasi-Frobenius algebra. Indeed, since $q_s \neq 0$ for all $s \in S$, $\mathscr{H}_k$ even is a symmetric algebra with respect to the trace form $\tau \colon \mathscr{H}_k \to k$ defined by $\tau(T_1) = 1$ and $\tau(T_w) = 0$ for $w \neq 1$; see, e.g., [13, 8.1.1].

It was first observed in [9, §2] that, in this situation, the results of Green [15] apply (the original applications of which have been to representations of G over fields of characteristic equal to p). Let us denote by $\mathrm{Irr}_k(G)$ the set of all simple kG-modules (up to isomorphism) and by $\mathrm{Irr}_k(G \mid B)$ the set of all $Y \in \mathrm{Irr}_k(G)$ such that Y is isomorphic to a submodule of $kG\underline{\mathfrak{b}}$. In the general framework of [19], this is the Harish-Chandra series consisting of the *unipotent principal series representations* of G. Furthermore, let $\mathrm{Irr}(\mathscr{H}_k)$ be the set of all simple $\mathscr{H}_k$-modules (up to isomorphism). Then, by [15, Theorem 2], the Hom functor $\mathfrak{F}_k$ induces a bijection

$$(\spadesuit) \qquad \mathrm{Irr}_k(G \mid B) \xrightarrow{\sim} \mathrm{Irr}(\mathscr{H}_k), \qquad M \mapsto \mathfrak{F}_k(M) = \mathrm{Hom}_{kG}(kG\underline{\mathfrak{b}}, M);$$

furthermore, by [15, Theorem 1], each indecomposable direct summand of $kG\underline{\mathfrak{b}}$ has a simple socle and a unique simple quotient. Combined with Remark 2.4, this already shows that St_k has a simple socle. More precisely, we have:

Theorem 3.1 (Cf. Tinberg [33, 4.10])**.** *Let $Y \subseteq \mathrm{St}_k$ be a simple submodule. Then $\underline{\mathfrak{u}}_1\mathfrak{e} \in Y$ and, hence, Y is uniquely determined. Furthermore, $\dim \mathfrak{F}_k(Y) = 1$ and the action of $\mathscr{H}_k$ on $\mathfrak{F}_k(Y)$ is given by the algebra homomorphism $\varepsilon \colon \mathscr{H}_k \to k$, $T_w \mapsto (-1)^{l(w)}$.*

Proof. By composing any map in $\mathfrak{F}_k(Y)$ with the inclusion $Y \subseteq \mathrm{St}_k$, we obtain an embedding $\mathfrak{F}_k(Y) \hookrightarrow \mathfrak{F}_k(\mathrm{St}_k)$ and we identify $\mathfrak{F}_k(Y)$ with a subset of $\mathfrak{F}_k(\mathrm{St}_k)$ in this way. Now $Y \subseteq \mathrm{St}_k \subseteq kG\underline{\mathfrak{b}}$ and so $\mathfrak{F}_k(Y) \neq \{0\}$ by Property (A) above. Consequently, by Corollary 2.3, we have $\mathfrak{F}_k(Y) = \mathfrak{F}_k(\mathrm{St}_k) = \langle \theta_{\underline{\mathfrak{u}}_1\mathfrak{e}} \rangle_k$ and $\mathscr{H}_k$ acts via ε. Furthermore, by the identification $\mathfrak{F}_k(Y) \subseteq \mathfrak{F}_k(\mathrm{St}_k)$, we must have $\theta_{\underline{\mathfrak{u}}_1\mathfrak{e}}(kG\underline{\mathfrak{b}}) \subseteq Y$ and so $\underline{\mathfrak{u}}_1\mathfrak{e} \in Y$. $\square$

Proposition 3.2. *Let $Y \subseteq \mathrm{St}_k$ be as in Theorem 3.1. Then Y is absolutely irreducible and occurs only once as a composition factor of St_k. Moreover, Y is the only composition factor of St_k which belongs to $\mathrm{Irr}_k(G \mid B)$.*

Proof. Recall that $\mathfrak{F}_k(Y) \neq \{0\}$ and Y corresponds to $\varepsilon\colon \mathscr{H}_k \to k$ via (♠). Hence, by [7, 2.13(d)], we have $\mathrm{End}_{kG}(Y) \cong \mathrm{End}_{\mathscr{H}_k}(\varepsilon) \cong k$ and so Y is absolutely irreducible. Now let $\{0\} = Z_0 \subsetneqq Z_1 \subsetneqq \ldots \subsetneqq Z_r = \mathrm{St}_k$ be a composition series and $Y_i := Z_i/Z_{i-1}$ for $i = 1, \ldots, r$ be the corresponding simple factors. Since $\ell \neq p$, the restriction of St_k to the subgroup $U \subseteq B$ is semisimple and, hence, isomorphic to the direct sum of the restrictions of the Y_i. Taking fixed points under U, we obtain

$$\dim \mathrm{Fix}_U(Y_1) + \ldots + \dim \mathrm{Fix}_U(Y_r) = \dim \mathrm{Fix}_U(\mathrm{St}_k) = \dim\langle \underline{\mathfrak{u}}_1 e\rangle_k = 1.$$

Now, if $Y_i \in \mathrm{Irr}_k(G \mid B)$, then $\mathrm{Fix}_U(Y_i) \supseteq \mathrm{Fix}_B(Y_i) \cong \mathfrak{F}_k(Y_i) \neq \{0\}$ by Property (A) and so we obtain a non-zero contribution to the sum on the left hand side. Hence, there can be at most one Y_i which belongs to $\mathrm{Irr}_k(G \mid B)$. Since $Y_1 = Y \subseteq kG\underline{\mathfrak{b}}$ does belong to $\mathrm{Irr}_k(G \mid B)$, this proves the remaining assertions. □

Example 3.3. It is easily seen that $\mathfrak{F}_k(k_G)$ is also 1-dimensional (spanned by the function $kG\underline{\mathfrak{b}} \to k$ which takes constant value 1 on all $g\underline{\mathfrak{b}}$ for $g \in G$) and $\mathscr{H}_k$ acts on $\mathfrak{F}_k(k_G)$ via the algebra homomorphism $\mathrm{ind}\colon \mathscr{H}_k \to k$ such that $\mathrm{ind}(T_s) = q_s$ for all $s \in S$; see, e.g., [12, 4.3.3]. Let Y be the simple socle of St_k, as in Theorem 3.1. Then, by (♠), we obtain:

$$Y \cong k_G \quad \Leftrightarrow \quad \mathfrak{F}_k(Y) \cong \mathfrak{F}_k(k_G) \quad \Leftrightarrow \quad \varepsilon = \mathrm{ind} \quad \Leftrightarrow \quad q_s = -1 \text{ for all } s \in S.$$

Thus, we recover a result of Hiss [18] and Khammash [26] in this way. Furthermore, Proposition 3.2 implies that, if $q_s \neq 1$ for some $s \in S$, then k_G is not even a composition factor of St_G. (This result is also contained in [18].)

Lemma 3.4. *Let M' be the Steinberg component in a given direct sum decomposition of $kG\underline{\mathfrak{b}}$, as in Remark 2.4. Then $\mathrm{St}_k = M'$ if and only if St_k is irreducible.*

Proof. Assume first that $M' = \mathrm{St}_k$ and let $Z \subsetneqq \mathrm{St}_k$ be a maximal submodule. Now $\mathrm{St}_k = M'$ is a factor module of $kG\underline{\mathfrak{b}}$ and so St_k/Z belongs to $\mathrm{Irr}_k(G \mid B)$, by Property (A). Hence, by Proposition 3.2, we must have $Z = \{0\}$. Conversely, assume that St_k is irreducible. Then $\ell \nmid [G : B]$ by Theorem 2.1. If $\ell = p$, then $\mathfrak{e}$ is a non-zero scalar multiple of an idempotent in kG, by [30, Lemma 2]. Hence, St_k is projective in this case and so St_k is a direct summand of $kG\underline{\mathfrak{b}}$. If $\ell \neq p$, then the assumption $\ell \nmid [G : B]$ implies that $kG\underline{\mathfrak{b}}$ is semisimple; see [12, Lemma 4.3.2]. Hence, again, St_k is a direct summand of $kG\underline{\mathfrak{b}}$. In both cases, it follows that $\mathrm{St}_k = M'$. □

Example 3.5. Assume that G has a BN-pair of rank 1, that is, $W = \langle s\rangle$ where $s \in W$ has order 2. Then, by the sharp form of the Bruhat decomposition, we have $[G : B] = 1 + |U| = 1 + \dim \mathrm{St}_k$. Thus, there are only two cases.

If $q_s \neq -1$, then $kG\underline{\mathfrak{b}} \cong k_G \oplus \mathrm{St}_k$ and St_k is irreducible by Theorem 2.1.

If $q_s = -1$, then the socle of St_k is the trivial module k_G by Example 3.3.

In the second case, the structure of St_k can be quite complicated. For example, let $G = {}^2G_2(q^2)$ be a Ree group, where q is an odd power of $\sqrt{3}$. If $\ell = 2$, then Landrock–Michler [27, Prop. 3.8(b)] determined socle series for $kG\underline{\mathfrak{b}}$ and St_k:

$$kG\underline{\mathfrak{b}}: \quad \begin{array}{ccc} & k_G & \\ & \varphi_2 & \\ \varphi_4 & \varphi_3 & \varphi_5 \\ & \varphi_2 & \\ & k_G & \end{array} \qquad\qquad \mathrm{St}_k: \quad \begin{array}{ccc} & \varphi_2 & \\ \varphi_4 & \varphi_3 & \varphi_5 \\ & \varphi_2 & \\ & k_G & \end{array}$$

where φ_i $(i = 1, 2, 3, 4, 5)$ are simple kG-modules and φ_4 is the contragredient dual of φ_5.

It is not true in general that St_k has a unique simple quotient. For example, let $G = \mathrm{GU}_3(q)$ where q is any prime power. Assume that ℓ is a prime such that $\ell \mid q+1$. Then socle series for St_k are known by the work of various authors; see Hiss [20, Theorem 4.1] and the references there:

$$\mathrm{St}_k: \qquad \begin{array}{c} \varphi \oplus \vartheta \\ k_G \\[1ex] (\ell = 2 \text{ and } 4 \mid q-1) \end{array} \qquad\qquad \begin{array}{c} \varphi \\ \vartheta \\ \varphi \\ k_G \\[1ex] (\text{otherwise}) \end{array}$$

where φ and ϑ are simple kG-modules. See also the examples in Gow [14, §5].

Example 3.6. Let $G = \mathrm{GL}_n(q)$ and $\mathscr{U}_k(G)$ be the set of all $Y \in \mathrm{Irr}_k(G)$ such that Y is a composition factor of $kG\underline{\mathfrak{b}}$. James [23, 16.4] called these the *unipotent modules* of G and showed that there is a canonical parametrization

$$\mathscr{U}_k(G) = \{D_\mu \mid \mu \vdash n\}.$$

(See also [24, 7.35].) Here, $D_{(n)} = k_G$, as follows immediately from [23, Def. 1.11].

The above parametrisation is characterised as follows. For each partition $\lambda \vdash n$, let M_λ be the permutation representation of G on the cosets of the corresponding parabolic subgroup $P_\lambda \subseteq G$ (block triangular matrices with diagonal blocks of sizes given by the parts of λ). Then D_μ has composition multiplicity 1 in M_μ and composition multipliciy 0 in M_λ unless $\lambda \trianglelefteq \mu$; see [23, 11.12(iv), 11.13]. This shows, in particular, that the above parametrisation is consistent with other known parametrisations of $\mathscr{U}_k(G)$, e.g., the one in [9, §3] based on properties of the ℓ-modular decomposition matrix of G.

If $\ell = 0$, let us set $e := \infty$; if ℓ is a prime ($\neq p$), then let

$$e := \min\{i \geqslant 2 \mid 1 + q + q^2 + \cdots + q^{i-1} \equiv 0 \bmod \ell\}.$$

Then, by James [24, Theorem 8.1(ix), (xi)], the subset $\mathrm{Irr}_k(G \mid B) \subseteq \mathscr{U}_k(G)$ consists precisely of those D_λ where $\lambda \vdash n$ is e-regular. Now let Y be the socle of St_k, as in Theorem 3.1. Then $Y \in \mathrm{Irr}_k(G \mid B)$ and so $Y \cong D_{\mu_0}$ for a well-defined e-regular partition $\mu_0 \vdash n$. This partition μ_0 can be identified as follows. Write $n = (e-1)m + r$ where

$0 \leqslant r < e-1$. (If $e = \infty$, then $m = 0$ and $r = n$.) We claim that

$$\mu_0 = (\underbrace{m+1, m+1, \ldots, m+1}_{r \text{ times}}, \underbrace{m, m, \ldots, m}_{e-r-1 \text{ times}}) \vdash n.$$

Indeed, by Theorem 3.1 and (♠), the kG-module Y corresponds to the 1-dimensional representation $\varepsilon\colon \mathscr{H}_k \to k$. Now $\mathrm{Irr}(\mathscr{H}_k)$ also has a natural parametrisation by the e-regular partitions of n, a result originally due to Dipper and James; see, e.g., [24, 8.1(i)], [12, §3.5] and the references there. By [24, Theorem 8.1(xii)] (or the general discussion in 3.7 below), this parametrisation is compatible with the above parametrisation of $\mathscr{U}_k(G)$, in the sense that the partition $\mu_0 \vdash n$ such that $Y \cong D_{\mu_0}$ must also parametrise ε. Now note that $\varepsilon \circ \gamma = \mathrm{ind}$, where $\mathrm{ind}\colon \mathscr{H}_k \to k$ is defined in Example 3.3 and $\gamma\colon \mathscr{H}_k \to \mathscr{H}_k$ is the algebra automorphism such that $\gamma(T_s) = -q_s T_s^{-1}$ (see [13, Exc. 8.2]). The definitions immediately show that ind is parametrised by the partition (n). Thus, our problem is a special case of describing the "Mullineux involution" on e-regular partitions which, for the particular partition (n), has the solution stated above by Mathas [28, 6.43(iii)]. (I thank Nicolas Jacon for pointing out this reference to me.)

We remark that Ackermann [1, Prop. 3.1] already showed that St_k has precisely one composition factor D_{μ_0} where μ_0 is the image of (1^n) under the Mullineux involution; however, he does not locate D_{μ_0} in a composition series of St_k.

3.7. For general G, the definition of *unipotent modules* is more complicated than for $\mathrm{GL}_n(q)$ (see, e.g., [10, §1]), but one can still proceed as follows. Let us assume that G is a true finite group of Lie type, as in Remark 2.5. We shall write $\mathrm{Irr}_{\mathbb{C}}(W) = \{E^\lambda \mid \lambda \in \Lambda\}$ where Λ is some finite indexing set. It is a classical fact that, if $k = \mathbb{C}$, then there is a bijection $\mathrm{Irr}_{\mathbb{C}}(W) \leftrightarrow \mathrm{Irr}(\mathscr{H}_{\mathbb{C}})$, $E^\lambda \leftrightarrow E_q^\lambda$, and a decomposition

$$\mathbb{C}G\underline{\mathfrak{b}} \cong \bigoplus_{\lambda \in \Lambda} \underbrace{\rho^\lambda \oplus \ldots \oplus \rho^\lambda}_{\dim E^\lambda \text{ times}} \qquad \text{where} \qquad \mathfrak{F}_{\mathbb{C}}(\rho^\lambda) \cong E_q^\lambda \quad \text{for all } \lambda \in \Lambda.$$

Hence, we have a natural parametrisation $\mathrm{Irr}_{\mathbb{C}}(G \mid B) = \{\rho^\lambda \mid \lambda \in \Lambda\}$ in this case; see, e.g., Carter [2, §10.11], Curtis–Reiner [4, §68B] (and also [7, Exp. 2.2], where the Hom functor is linked to the settings in [*loc. cit.*]). In general, under some mild conditions on k, it is shown in [8, Theorem 1.1] that there is still a natural parametrisation of $\mathrm{Irr}_k(G \mid B)$, but now by a certain subset $\Lambda_k^\circ \subseteq \Lambda$. We briefly describe how this is done, where we refer to the exposition in [12, §4.4] for further details and references.

First, to each E^λ one can attach a numerical value $\mathbf{a}_\lambda \in \mathbb{Z}_{\geqslant 0}$ (Lusztig's "$\mathbf{a}$-invariant"); note that $\lambda \mapsto \mathbf{a}_\lambda$ depends on the exponents c_s such that $|U_s| = q^{c_s}$ for $s \in S$. Then, under some mild conditions on k, the algebra $\mathscr{H}_k$ is "cellular" in the sense of Graham–Lehrer, where the corresponding cell modules are parametrized by Λ, and Λ is endowed with the partial order $\preceq$ such that $\mu \preceq \lambda$ if and only if $\mu = \lambda$ or $\mathbf{a}_\lambda < \mathbf{a}_\mu$. Finally, by the general theory of cellular algebras, there is a canonically defined subset $\Lambda_k^\circ \subseteq \Lambda$ such that

$$\mathrm{Irr}(\mathscr{H}_k) = \{L_k^\mu \mid \mu \in \Lambda_k^\circ\},$$

where L_k^λ is the unique simple quotient of the cell module corresponding to $\lambda \in \Lambda_k^\circ$. Hence, via the Hom functor and (♠), we obtain the desired parametrisation

$$\operatorname{Irr}_k(G \mid B) = \{Y^\mu \mid \mu \in \Lambda_k^\circ\} \qquad \text{where} \qquad \mathfrak{F}_k(Y^\mu) \cong L_k^\mu \text{ for } \mu \in \Lambda_k^\circ.$$

Let $M \in \operatorname{Irr}(\mathscr{H}_k)$ and denote by $d_{\lambda,M}$ the multiplicity of M as a composition factor of the cell module indexed by $\lambda \in \Lambda$. Then, by [12, 3.2.7], the unique $\mu \in \Lambda_k^\circ$ such that $M \cong L^\mu$ is characterised by the condition that μ is the unique element at which the function $\{\lambda \in \Lambda \mid d_{\lambda,M} \neq 0\} \to \mathbf{a}_\lambda$ takes its minimum value.

Now recall that the simple socle $Y \subseteq \operatorname{St}_k$ belongs to $\operatorname{Irr}_k(G \mid B)$ and it corresponds, via the Hom functor and (♠), to the 1-dimensional representation $\varepsilon\colon \mathscr{H}_k \to k$. The unique $\mu_0 \in \Lambda_k^\circ$ such that $Y \cong Y^{\mu_0}$ is found as follows. We order the elements of Λ according to increasing $\mathbf{a}$-invariant; then μ_0 is the first element in this list for which we have $d_{\mu_0,\varepsilon} \neq 0$. Note also that ε is afforded by a cell module; the unique $\lambda_0 \in \Lambda$ labelling this cell module is characterised by the condition that $\mathbf{a}_{\lambda_0} = \max\{\mathbf{a}_\lambda \mid \lambda \in \Lambda\}$ (see, e.g., [12, 1.3.3]).

For example, if $G = \operatorname{GL}_n(q)$, then $W = \mathfrak{S}_n$ and Λ is the set of partitions of n. In this case, we have $\lambda_0 = (1^n)$ and μ_0 is described in Example 3.6.

If tables with the decomposition numbers $d_{\lambda,M}$ for $\mathscr{H}_k$ are known, then μ_0 can be simply read off these tables. Thus, $\mu_0 \in \Lambda_k^\circ$ can be determined for all groups of exceptional type, using the information in [13, App. F], [12, Chap. 7]; the results are given in Table 1. (If there is no entry in this table corresponding to a certain value of e, then this means that $\ell \nmid [G : B]$ and so St_k is simple.)

Table 1. The labels $\mu_0 \in \Lambda_k^\circ$ for G of exceptional type and $\ell \mid [G : B]$

e	2	3	4	5	6	7	8	9	10	12	14	15	18	20	24	30
$G_2(q)$	1_W	σ_2			σ_1											
${}^3D_4(q)$	1_W	σ_2			ε_1					σ_1						
${}^2F_4(q^2)$	σ_2		ε_1		σ_2					σ_1						
$F_4(q)$	1_1	4_1	6_1		12		9_4			4_5						
${}^2E_6(q)$	8_1	4_1	1_3		4_4		8_4		8_2	9_4			4_5			
$E_6(q)$	1_p	15_q	10_s	$24_p'$	$60_p'$		$30_p'$	$20_p'$		$6_p'$						
$E_7(q)$	1_a	$15_a'$	$70_a'$	$84_a'$	$210_b'$	$27_a'$	$105_b'$	$35_b'$	21_b	56_a	$27_a'$		7_a			
$E_8(q)$	1_x	50_x	175_x	168_y	420_y	$300_x'$	$2835_x'$	$50_x'$	$448_z'$	$1400_x'$	$700_x'$	$84_x'$	$210_x'$	$112_z'$	$35_x'$	$8_z'$

In type E_8, $\ell > 5$; otherwise, $\ell > 3$; here, $e := \min\{i \geqslant 2 \mid 1 + q_0 + q_0^2 + \cdots + q_0^{i-1} \equiv 0 \bmod \ell\}$, with $q_0 := q$ in all cases except for ${}^2F_4(q^2)$, where q is an odd power of $\sqrt{2}$ and $q_0 := q^2$.

Just to illustrate the procedure (and to fix some notation), let us consider the case where $G = {}^2F_4(q^2)$; here, q is an odd power of $\sqrt{2}$. Setting $q_0 := q^2$, we have

$$|B| = q_0^{12}(q_0 - 1)^2 \qquad \text{and} \qquad [G : B] = (q_0 + 1)(q_0^2 + 1)(q_0^3 + 1)(q_0^6 + 1).$$

Now, $W = \langle s_1, s_2\rangle$ is dihedral of order 16 and we have $\{q_{s_1}, q_{s_2}\} = \{q_0, q_0^2\}$. We fix the notation such that $q_{s_1} = q_0^2$ and $q_{s_2} = q_0$. As in [12, Exp. 1.3.7], we have

$$\mathrm{Irr}_{\mathbb{C}}(W) = \{1_W, \varepsilon, \varepsilon_1, \varepsilon_2, \sigma_1, \sigma_2, \sigma_3\}$$

where ε_1, ε_2 are 1-dimensional and determined by $\varepsilon_1(s_1) = \varepsilon_2(s_2) = 1$ and $\varepsilon_1(s_2) = \varepsilon_2(s_1) = -1$; furthermore, each σ_i is 2-dimensional and the labelling is such that the trace of σ_1 on s_1s_2 equals $\sqrt{2}$, that of σ_2 equals 0, and that of σ_3 equals $-\sqrt{2}$.

The "mild condition" on k is that ℓ must be a "good" prime for the underlying algebraic group; so, $\ell > 3$. Assuming that $\ell \mid [G : B]$, we have the following cases to consider:

$$\ell \mid q_0 + 1, \qquad \ell \mid q_0^2 + 1, \qquad \ell \mid q_0^2 - q_0 + 1, \qquad \ell \mid q_0^4 - q_0^2 + 1,$$

which correspond to $e = 2, 4, 6, 12$, respectively. For example, for $e = 2, 4$, the decomposition numbers $d_{\lambda,M}$ are given as follows; see [12, Table 7.6]:

$(e=2)$	$\mathbf{a}_\lambda$	$d_{\lambda,M}$			
$\bullet\ 1_W$	0	1	.	.	.
ε_1	1	1	.	.	.
$\bullet\ \sigma_1$	2	.	.	1	.
$\bullet\ \sigma_2$	2	1	1	.	.
$\bullet\ \sigma_3$	2	.	.	.	1
ε_2	5	.	1	.	.
ε	12	.	1	.	.

$(e=4)$	$\mathbf{a}_\lambda$	$d_{\lambda,M}$			
$\bullet\ 1_W$	0	1	.	.	.
$\bullet\ \varepsilon_1$	1	.	1	.	.
$\bullet\ \sigma_1$	2	.	.	1	.
$\bullet\ \sigma_2$	2	.	.	.	1
σ_3	2	1	1	.	.
ε_2	5	1	.	.	.
ε	12	.	1	.	.

Those representations which belong to Λ_k° are marked by "$\bullet$". The above procedure for finding μ_0 now yields $\mu_0 = \sigma_2$ for $e = 2$ and $\mu_0 = \varepsilon_1$ for $e = 4$.

Remark 3.8. For groups of classical type, Λ is a set of certain bipartitions of n and the subsets $\Lambda_k^\circ \subseteq \Lambda$ are explicitly known in all cases; see [12]. Nicolas Jacon has pointed out to me that then $\mu_0 \in \Lambda_k^\circ$ can be described explicitly using the results of Jacon–Lecouvey [22]. This will be discussed elsewhere in more detail.

4. The Steinberg module and Harish-Chandra series

We shall assume from now on that $G = \mathbf{G}^F$ is a true finite group of Lie type, as in Remark 2.5. Then G satisfies the "commutator relations" and so the parabolic subgroups of G admit Levi decompositions; see Carter [2, §2.6], Curtis–Reiner [4, §70A]. For each subset $J \subseteq S$, let $P_J \subseteq G$ be the corresponding standard parabolic subgroup, with Levi decomposition $P_J = U_J \rtimes L_J$ where U_J is the unipotent radical of P_J and L_J is the standard Levi complement. The Weyl group of L_J is $W_J = \langle J\rangle$ and L_J itself is a true finite group of Lie type. Let A be a commutative ring (with 1_A) such that p is invertible in A. Then we obtain functors, called Harish-Chandra induction and restriction,

$$R_J^S\colon AL_J\text{-modules} \to AG\text{-modules},$$

$${}^*R_J^S\colon AG\text{-modules} \to AL_J\text{-modules},$$

which satisfy properties analogous to the usual induction and restriction, like transitivity, adjointness and a Mackey formula; we refer to [5], [19] and the survey in [7, §3] for further details. An AG-module Y is called *cuspidal* if ${}^*R_J^S(Y)=\{0\}$ for all $J \subsetneqq S$. In this general setting, we have the following important result due to Dipper–Du [5] and Howlett–Lehrer [21]. Let $I,J\subseteq S$ be subsets such that $wIw^{-1}=J$ for some $w\in W$; let $n\in N$ be a representative of w. Then $nL_In^{-1}=L_J$ and

$$R_I^S(X)\cong R_J^S({}^nX) \qquad \text{for any } AL_I\text{-module } X.$$

(Here, we denote by ${}^nX$ the usual conjugate module for $\mathscr{O}L_J$; see, e.g., [3, §10B]). In analogy to [4, (70.11)], we will refer to this as the "Strong Conjugacy Theorem".

Furthermore, we now place ourselves in the usual setting for modular representation theory; see, e.g., [3, §16A]. Thus, we assume that our field k has characteristic $\ell>0$ (where $\ell\neq p$ as before), and that k is the residue field of a discrete valuation ring $\mathscr{O}$ with field of fractions K of characteristic 0. Both K and k will be assumed to be "large enough", that is, K and k are splitting fields for G and all its subgroups. An $\mathscr{O}G$-module M which is finitely generated and free over $\mathscr{O}$ will be called an $\mathscr{O}G$-lattice. If M is an $\mathscr{O}G$-lattice, then we may naturally regard M as a subset of the KG-module $KM:=K\otimes_{\mathscr{O}}M$; furthermore, by "$\ell$-modular reduction", we obtain a kG-module $\overline{M}:=M/\mathfrak{p}M\cong k\otimes_{\mathscr{O}}M$ where $\mathfrak{p}$ is the unique maximal ideal of $\mathscr{O}$. Finally note that, by [3, Exc. 6.16], idempotents can be lifted from kG to $\mathscr{O}G$, hence, $\mathscr{O}G$ is "semiperfect". We shall freely use standard notions and properties of projective covers, pure submodules etc.; see [3, §4D, §6].

Harish-Chandra induction and restriction are compatible with this set-up. Indeed, let $J\subseteq S$. If X is an $\mathscr{O}L_J$-lattice and Y is an $\mathscr{O}G$-lattice, then $R_J^S(X)$ is an $\mathscr{O}G$-lattice, ${}^*R_J^S(Y)$ is an $\mathscr{O}L_J$-lattice, and we have

$$\begin{aligned} KR_J^S(X)&\cong R_J^S(KX) &\qquad \text{and}\qquad && K{}^*R_J^S(Y)&\cong {}^*R_J^S(KY),\\ \overline{R_J^S(X)}&\cong R_J^S(\overline{X}) &\qquad \text{and}\qquad && \overline{{}^*R_J^S(Y)}&\cong {}^*R_J^S(\overline{Y}).\end{aligned}$$

4.1. By Theorem 2.1, we have the "canonical" Steinberg lattice $\mathrm{St}_{\mathscr{O}}=\mathscr{O}G\mathfrak{e}$. Here, we can naturally identify $K\mathrm{St}_{\mathscr{O}}$ with St_K and $\overline{\mathrm{St}}_{\mathscr{O}}$ with St_k. Since $\mathrm{char}(K)=0$, the KG-module St_K is irreducible. We obtain further $\mathscr{O}G$-lattices affording St_K as follows. Let $\sigma\colon U\to K^\times$ be a group homomorphism. Since $\ell\nmid |U|$, the values of σ lie in $\mathscr{O}^\times$. Then $\underline{\mathfrak{u}}_\sigma:=\sum_{u\in U}\sigma(u)u\in\mathscr{O}G$ and so $\Gamma_\sigma:=\mathscr{O}G\underline{\mathfrak{u}}_\sigma$ is an $\mathscr{O}G$-lattice. Since $\underline{\mathfrak{u}}_\sigma^2=|U|\underline{\mathfrak{u}}_\sigma$ and $|U|$ is a unit in $\mathscr{O}$, the module Γ_σ is projective. Furthermore, $\mathrm{Hom}_{\mathscr{O}G}(\Gamma_\sigma,\mathrm{St}_{\mathscr{O}})\cong\underline{\mathfrak{u}}_\sigma\mathrm{St}_{\mathscr{O}}=\langle\underline{\mathfrak{u}}_\sigma\mathfrak{e}\rangle_{\mathscr{O}}$ where the last equality holds since $\{u\mathfrak{e}\mid u\in U\}$ is an $\mathscr{O}$-basis of $\mathrm{St}_{\mathscr{O}}$ and since $\underline{\mathfrak{u}}_\sigma u=\sigma(u)^{-1}\underline{\mathfrak{u}}_\sigma$ for all $u\in U$. Thus,

$$\mathrm{Hom}_{\mathscr{O}G}(\Gamma_\sigma,\mathrm{St}_{\mathscr{O}})=\langle\varphi_\sigma\rangle_{\mathscr{O}} \qquad\text{where}\qquad \varphi_\sigma\colon\Gamma_\sigma\to\mathrm{St}_{\mathscr{O}},\ \ \gamma\mapsto\gamma\underline{\mathfrak{u}}_\sigma\mathfrak{e}.$$

The same computation also works over K, hence we obtain $\dim\mathrm{Hom}_{KG}(K\Gamma_\sigma,\mathrm{St}_K)=1$.

Proposition 4.2 (Cf. Hiss [19, §6])**.** *For any $\sigma\colon U\to K^\times$ as above, there is a unique pure $\mathscr{O}G$-sublattice $\Gamma_\sigma'\subseteq\Gamma_\sigma$ such that, if we set $\mathscr{S}_\sigma:=\Gamma_\sigma/\Gamma_\sigma'$, then $K\mathscr{S}_\sigma\cong\mathrm{St}_K$. Furthermore,*

φ_σ induces an injective $\mathscr{O}G$-module homomorphism $\rho_\sigma\colon \mathscr{S}_\sigma \to \mathrm{St}_{\mathscr{O}}$. The kG-module $D_\sigma := \overline{\mathscr{S}}_\sigma/\mathrm{rad}(\overline{\mathscr{S}}_\sigma)$ is simple and it occurs exactly once as a composition factor of St_k.

Proof. First we show that a sublattice $\Gamma'_\sigma \subseteq \Gamma_\sigma$ with the desired properties exists. Now, since KG is semisimple and $\dim \mathrm{Hom}_{KG}(K\Gamma_\sigma, \mathrm{St}_K) = 1$, we can write $K\Gamma_\sigma = Z_1 \oplus Z_2$ where Z_1, Z_2 are KG-submodules such that $Z_1 \cong \mathrm{St}_K$ and $\mathrm{Hom}_{KG}(Z_2, \mathrm{St}_K) = \{0\}$. Then $\Gamma'_\sigma := \Gamma_\sigma \cap Z_2$ is a pure submodule of Γ_σ. Consequently, $\mathscr{S}_\sigma := \Gamma_\sigma/\Gamma'_\sigma$ is an $\mathscr{O}G$-lattice such that $K\mathscr{S}_\sigma \cong \mathrm{St}_K$. Now consider the map $\varphi_\sigma\colon \Gamma_\sigma \to \mathrm{St}_{\mathscr{O}}$. Since $\mathrm{Hom}_{KG}(Z_2, \mathrm{St}_K) = \{0\}$, we have $\Gamma'_\sigma \subseteq \ker(\varphi_\sigma)$ and so we obtain an induced $\mathscr{O}G$-module homomorphism $\rho_\sigma\colon \mathscr{S}_\sigma \to \mathrm{St}_{\mathscr{O}}$. Since $K\mathscr{S}_\sigma$ is irreducible and $\varphi_\sigma \neq 0$, the map ρ_σ is injective.

Let us further write $\Gamma_\sigma = P_1 \oplus \ldots \oplus P_r$ where each P_i is an $\mathscr{O}G$-lattice which is projective and indecomposable. Then $K\Gamma_\sigma = KP_1 \oplus \ldots \oplus KP_r$. Since $\dim \mathrm{Hom}_{KG}(K\Gamma_\sigma, \mathrm{St}_K) = 1$, there is a unique i such that $\mathrm{Hom}_{KG}(KP_i, \mathrm{St}_K) \neq \{0\}$. Then $Z_1 \subseteq KP_i$ and $KP_j \subseteq Z_2$ for all $j \neq i$. Thus, we have $\mathscr{S}_\sigma \cong P_i/(P_i \cap Z_2)$ and so P_i is a projective cover of $\mathscr{S}_\sigma$. This certainly implies that $D_\sigma = \overline{\mathscr{S}}_\sigma/\mathrm{rad}(\overline{\mathscr{S}}_\sigma) \cong \overline{P}_i/\mathrm{rad}(\overline{P}_i)$ is simple and that $\overline{P}_i$ is a projective cover of D_σ. Since $\underline{\mathfrak{u}}_\sigma \mathfrak{e} \in \rho_\sigma(\mathscr{S}_\sigma)$, the induced map $\overline{\rho}_\sigma\colon \overline{\mathscr{S}}_\sigma \to \mathrm{St}_k$ is non-zero and so D_σ is a composition factor of St_k. On the other hand, by standard properties of projective modules, the composition multiplicity of D_σ in St_k is bounded above by

$$\dim \mathrm{Hom}_{kG}(P_i, \mathrm{St}_k) \leqslant \dim \mathrm{Hom}_{kG}(\overline{\Gamma}_\sigma, \mathrm{St}_k) = \mathrm{Hom}_{KG}(K\Gamma_\sigma, \mathrm{St}_K) = 1.$$

Once the existence of Γ'_σ is shown, the uniqueness automatically follows since the intersection of pure submodules is pure and St_K has multiplicity 1 in $K\Gamma_\sigma$. $\square$

4.3. We assume from now on that the center of $\mathbf{G}$ is connected. Furthermore, we shall fix a group homomorphism $\sigma\colon U \to K^\times$ which is a *regular character*, that is, we have $U^* \subseteq \ker(\sigma)$ and the restriction of σ to U_s is non-trivial for all $s \in S$. (Such characters always exist.) Then the corresponding module $\Gamma_\sigma = \mathscr{O}G\underline{\mathfrak{u}}_\sigma$ is called a *Gelfand-Graev module* for G; see [2, §8.1] or [31, p. 258]. Since the center of $\mathbf{G}$ is assumed to be connected, all regular characters of U are conjugate under the action of H and, hence, the corresponding Gelfand-Graev modules will all be isomorphic; see [2, 8.1.2].

For any $J \subseteq S$, we have $L_J = \mathbf{L}^F$ where $\mathbf{L}$ is an F-stable Levi subgroup in $\mathbf{G}$; here, the center of $\mathbf{L}$ will also be connected; see [2, 8.1.4]. Our regular character σ determines a regular character of L_J; see, e.g., [2, 8.1.6]. Hence, we also have a well-defined Gelfand-Graev module for $\mathscr{O}L_J$, which we denote by Γ_σ^J. Applying the construction in Proposition 4.2, we obtain an $\mathscr{O}L_J$-lattice $\mathscr{S}_\sigma^J = \Gamma_\sigma^J/(\Gamma_\sigma^J)'$ and a simple kL_J-module $D_\sigma^J := \overline{\mathscr{S}}_\sigma^J/\mathrm{rad}(\overline{\mathscr{S}}_\sigma^J)$. We have $K\mathscr{S}_\sigma^J \cong \mathrm{St}_K^J$, the Steinberg module for KL_J.

Lemma 4.4. *Let $J \subseteq S$. Then the following hold.*

(i) *We have ${}^*R_J^S(\Gamma_\sigma) \cong \Gamma_\sigma^J$ and ${}^*R_J^S(\mathscr{S}_\sigma) \cong \mathscr{S}_\sigma^J$ (as $\mathscr{O}L_J$-modules).*
(ii) *If $I \subseteq S$ and $n \in N$ are such that $nL_In^{-1} = L_J$, then ${}^n\mathscr{S}_\sigma^I \cong \mathscr{S}_\sigma^J$ (as $\mathscr{O}L_J$-modules) and ${}^nD_\sigma^I \cong D_\sigma^J$ (as kL_J-modules).*

Proof. (i) By a result of Rodier ([2, 8.1.5]), we have ${}^*R_J^S(K\Gamma_\sigma) \cong K\Gamma_\sigma^J$; by [4, (71.6)], we also have ${}^*R_J^S(\mathrm{St}_K) \cong \mathrm{St}_K^J$. So (i) follows by a standard argument; see, e.g., [7, 5.14, 5.15].

(ii) Since ${}^*R_J^S(K\Gamma_\sigma) \cong K\Gamma_\sigma^J$, it is straightforward to show that ${}^nK\Gamma_\sigma^I \cong K\Gamma_\sigma^J$, using the "Strong Conjugacy Theorem". So we also have ${}^n\Gamma_\sigma^I \cong \Gamma_\sigma^J$ as $\mathscr{O}L_J$-modules (since these modules are projective). This then implies (ii) by the construction of $\mathscr{S}_\sigma^J$ and D_σ^J. $\square$

4.5. Let $\mathscr{P}_\sigma^*$ be the set of all subsets $J \subseteq S$ such that D_σ^J is a cuspidal kL_J-module. For $J \in \mathscr{P}_\sigma^*$, we denote by $\mathrm{Irr}_k(G \mid J, \sigma)$ the set of all $Y \in \mathrm{Irr}_k(G)$ such that Y is isomorphic to a submodule of $R_J^S(D_\sigma^J)$. By the "Strong Conjugacy Theorem", this is a Harish-Chandra series as defined by Hiss [19]. Hence, by [19, Theorem 5.8] (see also [7, §3]), every simple submodule of $R_J^S(D_\sigma^J)$ is isomorphic to a factor module of $R_J^S(D_\sigma^J)$, and vice versa. Furthermore, using also Lemma 4.4(ii), we have for any $I, J \in \mathscr{P}_\sigma^*$:

$$\begin{aligned} &\mathrm{Irr}_k(G \mid I, \sigma) = \mathrm{Irr}_k(G \mid J, \sigma) && \text{if } J = wIw^{-1} \text{ for some } w \in W, \\ &\mathrm{Irr}_k(G \mid I, \sigma) \cap \mathrm{Irr}_k(G \mid J, \sigma) = \varnothing && \text{otherwise.} \end{aligned}$$

Thus, having fixed a regular character $\sigma \colon U \to K^\times$ as in 4.3, the above constructions produce composition factors of $kG\underline{\mathfrak{b}}$ arising from subsets $J \subseteq S$. The following two results are adaptations of Dipper–Gruber [6, Cor. 2.24 and 2.40] to the present setting.

Proposition 4.6. *Let $J \in \mathscr{P}_\sigma^*$. Then St_k has a unique composition factor which belongs to the series $\mathrm{Irr}_k(G \mid J, \sigma)$.*

Proof. First note that $\mathrm{St}_K \cong K\mathscr{S}_\sigma$ and so, by a classical result of Brauer–Nesbitt (see [3, (16.16)]), St_k and $\overline{\mathscr{S}}_\sigma$ have the same composition factors (counting multiplicities). Using Lemma 4.4(i) and adjointness, we obtain

$$\mathrm{Hom}_{kG}\big(\overline{\mathscr{S}}_\sigma, R_J^S(D_\sigma^J)\big) \cong \mathrm{Hom}_{kL_J}\big({}^*R_J^S(\overline{\mathscr{S}}_\sigma), D_\sigma^J\big) \cong \mathrm{Hom}_{kL_J}(\overline{\mathscr{S}}_\sigma^J, D_\sigma^J) \neq \{0\}.$$

Hence, some simple submodule of $R_J^S(D_\sigma^J)$ will be isomorphic to a composition factor of $\overline{\mathscr{S}}_\sigma$ and so the latter module has at least one composition factor which belongs to $\mathrm{Irr}_k(G \mid J, \sigma)$. On the other hand, since D_σ^J is a quotient of $\overline{\Gamma}_\sigma^J$, there exists a surjective kG-module homomorphism $R_J^S(\overline{\Gamma}_\sigma^J) \to R_J^S(D_\sigma^J)$. Now $R_J^S(\overline{\Gamma}_\sigma^J)$ is projective (see, e.g., [7, 3.4]) and every simple module in $\mathrm{Irr}_k(G \mid J, \sigma)$ also is a quotient of $R_J^S(D_\sigma^J)$ (see 4.5). Hence, by standard results on projective modules, the total number of composition factors (counting multiplicities) of $\overline{\mathscr{S}}_\sigma$ which belong to $\mathrm{Irr}_k(G \mid J, \sigma)$ is bounded above by

$$\dim \mathrm{Hom}_{kG}\big(R_J^S(\overline{\Gamma}_\sigma^J), \overline{\mathscr{S}}_\sigma\big) = \dim \mathrm{Hom}_{KG}\big(R_J^S(K\Gamma_\sigma^J), K\mathscr{S}_\sigma\big).$$

Using Lemma 4.4(i) and adjointness, the dimension on the right hand side evaluates to $\dim \mathrm{Hom}_{KL_J}(K\Gamma_\sigma^J, K\mathscr{S}_\sigma^J)$, which is one by 4.1. $\square$

Proposition 4.7. *Assume that every composition factor of $kG\underline{\mathfrak{b}}$ belongs to $\mathrm{Irr}_k(G \mid J, \sigma)$ for some $J \in \mathscr{P}_\sigma^*$. Then the following hold.*

(i) *St_k is multiplicity-free and the length of a composition series of St_k is equal to the number of $J \in \mathscr{P}_\sigma^*$ (up to W-conjugacy).*
(ii) *The induced map $\overline{\rho}_\sigma \colon \overline{\mathscr{S}}_\sigma \to \mathrm{St}_k$ is an isomorphism and so $\mathrm{St}_k/\mathrm{rad}(\mathrm{St}_k) \cong D_\sigma$.*
(iii) *All composition factors of $\mathrm{rad}(\mathrm{St}_k)$ are non-cuspidal.*

Proof. (i) Since $\mathrm{St}_k \subseteq kG\underline{\mathfrak{b}}$, the hypothesis applies, in particular, to the composition factors of St_k. It remains to use Proposition 4.6.

(ii) By the proof of Proposition 4.2, we have $\overline{\rho}_\sigma \neq 0$. Hence, it is enough to show that $\overline{\rho}_\sigma$ is injective. By [7, Theorem 5.16], this is equivalent to the following statement.

(†) If $I \subseteq S$ is such that $\overline{\mathscr{S}}_\sigma^I$ has a cuspidal simple submodule, then $I = \varnothing$.

Now (†) is proved as follows. Let $X \subseteq \overline{\mathscr{S}}_\sigma^I$ be a cuspidal simple submodule. Using Lemma 4.4(i) and adjointness, we obtain that

$$\mathrm{Hom}_{kG}(R_I^S(X), \overline{\mathscr{S}}_\sigma) \cong \mathrm{Hom}_{kL_I}(X, \overline{\mathscr{S}}_\sigma^I) \neq \{0\}.$$

Thus, $\overline{\mathscr{S}}_\sigma$ has a composition factor which is a quotient of $R_I^S(X)$. Since $\overline{\mathscr{S}}_\sigma$ and $\mathrm{St}_k \subseteq kG\underline{\mathfrak{b}}$ have the same composition factors, it follows that $kG\underline{\mathfrak{b}}$ has a composition factor which is a quotient of $R_I^S(X)$. By our assumption and the characterisation of Harish-Chandra series in [19, Theorem 5.8], the pair (I, X) is N-conjugate to a pair (J, D_σ^J) where $J \in \mathscr{P}_\sigma^*$. So there exists some $n \in N$ such that $nL_In^{-1} = L_J$ and ${}^nX \cong D_\sigma^J$. Using Lemma 4.4(ii), we conclude that $X \cong D_\sigma^I$. Thus, D_σ^I is both isomorphic to a submodule and to a quotient of $\overline{\mathscr{S}}_\sigma^I$. Now, having a unique simple quotient, the module $\overline{\mathscr{S}}_\sigma^I$ is indecomposable. Hence, the multiplicity 1 statement in Proposition 4.2 implies that $D_\sigma^I \cong \overline{\mathscr{S}}_\sigma^I$ and, consequently, we also have $D_\sigma^I \cong \mathrm{St}_k^I \subseteq kL_I\underline{\mathfrak{b}}_I$. Thus, $kL_I\underline{\mathfrak{b}}_I \cong R_\varnothing^I(k_H)$ has a cuspidal simple submodule. Again, by [19, Theorem 5.8], this can only happen if $I = \varnothing$.

(iii) By our assumption, the only composition factor of St_k which can possibly be cuspidal is D_σ. By (ii) and Proposition 4.2, D_σ is not a composition factor of $\mathrm{rad}(\mathrm{St}_k)$. ☐

Remark 4.8. In analogy to Example 3.6, we define $\mathscr{U}_k(G)$ to be the set of all $Y \in \mathrm{Irr}_k(G)$ which are composition factors of $kG\underline{\mathfrak{b}}$. We have $\mathrm{Irr}_k(G \mid B) \subseteq \mathscr{U}_k(G)$ but note that, in general, we neither have equality nor is $\mathscr{U}_k(G)$ the complete set of all *unipotent kG-modules* as defined, for example, in [10, §1]. (Over K, we do have at least $\mathrm{Irr}_K(G \mid B) = \mathscr{U}_K(G)$.) For $J \subseteq S$, we define $\mathscr{U}_k(L_J)$ analogously; the standard Borel subgroup of L_J is given by $B_J := B \cap L_J$. Let $X \in \mathscr{U}_k(L_J)$ and $Y \in \mathscr{U}_k(G)$. Then we have:

(a) All composition factors of $R_J^S(X)$ belong to $\mathscr{U}_k(G)$.

(b) If ${}^*R_J^S(Y) \neq \{0\}$, then all composition factors of ${}^*R_J^S(Y)$ belong to $\mathscr{U}_k(L_J)$.

Proof. (a) By the definitions, we have $kG\underline{\mathfrak{b}} \cong R_\varnothing^S(k_H)$ and, similarly, $kL_J\underline{\mathfrak{b}}_J \cong R_\varnothing^J(k_H)$, where $\underline{\mathfrak{b}}_J = \sum_{b \in B \cap L_J} b \in kL_J$. Hence, using transitivity, we obtain $kG\underline{\mathfrak{b}} \cong R_J^S(kL_J\underline{\mathfrak{b}}_J)$. Since Harish-Chandra induction is exact (see [7, 3.4]), $R_J^S(X)$ is a subquotient of $kG\underline{\mathfrak{b}}$.

(b) Since $kG\underline{\mathfrak{b}} \cong R_\varnothing^S(k_H)$, the Mackey formula immediately shows that ${}^*R_J^S(kG\underline{\mathfrak{b}})$ is a direct sum of a certain number of copies of $kL_J\underline{\mathfrak{b}}_J$. It remains to use the fact that Harish-Chandra restriction is also exact. ☐

Example 4.9. Let $G = \mathrm{GL}_n(q)$, where $n \geqslant 1$ and q is any prime power. Let $e \geqslant 2$ be defined as in Example 3.6; also recall that $|\mathscr{U}_k(G)| = \pi(n)$, where $\pi(n)$ denotes the number of partitions of n. By [10, 7.6], D_σ is cuspidal if and only if $n = 1$ or $n = e\ell^j$ for some $j \geqslant 0$. (Note that, if $\ell \mid q-1$, then our e equals ℓ, while it equals 1 in [*loc.*

cit.]; otherwise, the two definitions coincide.) Now, the W-conjugacy classes of subsets $J \subseteq S$ are parametrised by the partitions $\lambda \vdash n$ (see [13, 2.3.8]); the Levi subgroup L_J corresponding to λ is a direct product of general linear groups corresponding to the parts of λ. Hence, the subsets $J \in \mathscr{P}_\sigma^*$ are parametrized by the partitions $\lambda \vdash n$ such that each part of λ is equal to 1 or to $e\ell^j$ for some $j \geqslant 0$. So Remark 4.8(a) and the counting argument in [11, (2.5)] yield $\pi(n)$ simple modules in $\mathscr{U}_k(G)$ which belong to $\mathrm{Irr}_k(G \mid J, \sigma)$ for some $J \in \mathscr{P}_\sigma^*$. Thus, the hypothesis of Proposition 4.7 is satisfied in this case. Consequently, $\mathrm{St}_k/\mathrm{rad}(\mathrm{St}_k)$ is simple and St_k is multiplicity-free. (This was also shown by Szechtman [32], using different techniques.) Furthermore, the composition length of St_k is the coefficient of t^n in the power series

$$\frac{1}{1-t}\prod_{j\geqslant 0}\frac{1}{1-t^{e\ell^j}}.$$

Indeed, let c_n denote the composition length of St_k. By Proposition 4.7(i), c_n equals the number of $J \in \mathscr{P}_\sigma^*$ (up to W-conjugacy). By the above disussion (see also [11, (2.5)]), this is equal to the number of sequences $(m_{-1}, m_0, m_1, \ldots)$ of non-negative integers such that $m_{-1} + em_0 + e\ell m_1 + \cdots = n$ (where $\mathrm{GL}_0(q) = \{1\}$ by convention). We multiply both sides by t^n and sum over all $n \geqslant 0$. This yields

$$\sum_{n\geqslant 0} c_n t^n = \sum_{(m_{-1},m_0,m_1,\ldots)} t^{m_{-1}+em_0+e\ell m_1+\ldots} = \Big(\sum_{m_{-1}\geqslant 0} t^{m_{-1}}\Big)\Big(\sum_{m_0\geqslant 0} t^{em_0}\Big)\Big(\sum_{m_1\geqslant 0} t^{e\ell m_1}\Big)\cdots.$$

Using the identity $1/(1-t^r) = \sum_{i\geqslant 0} t^{ri}$ for all $r \geqslant 1$, we obtain the desired formula.

Remark 4.10. In the setting of Szechtman [32], the above expression for c_n means that the formula (4) in [32, p. 605] holds for all n. (Previously, this was only known for $n \leqslant 10$; see the remarks in [*loc. cit.*].) This formula gives an explicit expression of the layers in the Jantzen filtration of St_k (as defined by Gow [14]), as direct sums of simple modules. It also shows that the layers in this filtration are not always irreducible and, hence, Gow's conjecture [14, 6.3] does not hold in general. See also the explicit examples in [32, §9].

Example 4.11. Let $G = G_n(q)$, $n \geqslant 1$, be one of the following finite classical groups:

(1) The general unitary group $\mathrm{GU}_n(q)$ for any n, any q.
(2) The special orthogonal group $\mathrm{SO}_n(q)$ where $n = 2m+1$ is odd and q is odd.
(3a) The symplectic group $\mathrm{Sp}_n(q)$ where $n = 2m$ is even and q is a power of 2.
(3b) The conformal symplectic group $\mathrm{CSp}_n(q)$ where $n = 2m$ is even and q is odd.
(4) The conformal orthogonal group $\mathrm{CSO}_n^{\pm}(q)$ where $n = 2m$ is even and q is odd.

Each of these groups can be realized as $G = \mathbf{G}^F$ where $\mathbf{G}$ has a connected center and $\mathbf{G}$ is simple modulo its center. By convention, $G_0(q)$ is the trivial group, except for the conformal groups, where it is cyclic of order $q-1$. We define the parameter δ to be 2 in case (1) and to be 1 in all the remaining cases.

Theorem 4.12 ([9], Gruber [16], and Gruber–Hiss [17])**.** *Let $G = G_n(q)$ be as in Example 4.11 and assume that ℓ is "linear", that is, $q^{\delta m-1} \not\equiv -1 \bmod \ell$ for all $m \geqslant 1$.*

(i) *We have* $|\mathrm{Irr}_{\mathbb{C}}(W)| = |\mathscr{U}_k(G)|$.

(ii) *If* $Y \in \mathscr{U}_k(G)$, *then* ${}^*R_J^S(Y) \neq \{0\}$ *for some subset* $J \subseteq S$ *such that* L_J *is a direct product of groups of untwisted type* A.

Proof. This follows from [9, §4] in the cases (1), (2), (3a), (3b), and from [16] in case (4). We shall refer to the more general setting in [17] (where all of $\mathrm{Irr}_k(G)$ is considered).

(i) Note that, by the "classical fact" in characteristic 0 mentioned in 3.7, we certainly know that $|\mathrm{Irr}_{\mathbb{C}}(W)| = |\mathscr{U}_K(G)|$. Hence, the assertion immediately follows from the block diagonal shape of the decomposition matrix in [17, Theorem 8.2].

(ii) Let Q be a projective indecomposable $\mathscr{O}G$-lattice such that $\overline{Q}$ is a projective cover of Y. By [17, Cor. 8.6], Q is a direct summand of $R_J^S(Q')$, where $J \subseteq S$ and Q' is a projective indecomposable $\mathscr{O}L_J$-lattice such that the following conditions hold. First, we have $L_J \cong G_a(q) \times L_\lambda$ where $n = a + 2m$ $(a, m \geqslant 0)$ and $G_a(q)$ is a group of the same type as G; furthermore, λ is a composition of m and L_λ is a direct product of general linear groups $\mathrm{GL}_{\lambda_i}(q^\delta)$ where λ_i runs over the non-zero parts of λ. Finally, under the isomorphism $L_J \cong G_a(q) \times L_\lambda$, we have $Q' \cong Q'_a \otimes Q'_\lambda$ where Q'_a is a projective indecomposable $\mathscr{O}G_a(q)$-lattice such that KQ'_a has only cuspidal constituents and Q'_λ is an indecomposable direct summand of the Gelfand-Graev lattice for $\mathscr{O}L_\lambda$.

Now, since $\overline{Q}$ is a direct summand of $R_J^S(\overline{Q}')$, we have $\mathrm{Hom}_{kL_J}(\overline{Q}', {}^*R_J^S(Y)) \neq \{0\}$ by adjointness. This shows, first of all, that ${}^*R_J^S(Y) \neq \{0\}$. Using Remark 4.8(b), we conclude that $\mathrm{Hom}_{kL_J}(\overline{Q}', kL_J\underline{\mathfrak{b}}_J) \neq 0$. Consequently, since Q' is projective, we also have $\mathrm{Hom}_{KL_J}(KQ', KL_J\underline{\mathfrak{b}}_J) \neq 0$. So, by the above direct product decomposition of L_J, at least one of the cuspidal composition factors of KQ'_a belongs to $\mathscr{U}_K(G_a(q))$. But this can only happen if $G_a(q)$ has BN-rank equal to 0. Thus, L_J has the required form. $\square$

4.13. Let $G = G_n(q)$ be as in Example 4.11 and assume that ℓ is linear. By Theorem 4.12(ii) and the characterisation of Harish-Chandra series in [19, Theorem 5.8], every $Y \in \mathscr{U}_k(G)$ is a submodule of $R_J^S(X)$ where $J \subseteq S$ is such that L_J is isomorphic to a direct product of groups of untwisted type A, and $X \in \mathrm{Irr}_k(L_J)$ is cuspidal. Then, by adjointness, X is a composition factor of ${}^*R_J^S(Y)$ and, hence, $X \in \mathscr{U}_k(L_J)$ by Remark 4.8(b). But then the known results on general linear groups imply that $X \cong D_\sigma^J$; see, e.g., [7, Cor. 6.16]. Thus, the hypothesis of Proposition 4.7 is satisfied. (This is also mentioned in Dipper–Gruber [6, 4.22], with only a sketch proof.)

Thus, in all the cases listed in Example 4.11, St_k is multiplicity-free, $\mathrm{St}_k/\mathrm{rad}(\mathrm{St}_k)$ is simple and the composition length of St_k is determined as in Proposition 4.7(i). Consequently, one can derive a generating function for the composition length of St_k, similar to that for $\mathrm{GL}_n(q)$ in Example 4.9. We will only give the details for $G = \mathrm{GU}_n(q)$. Write $n = 2m$ (if n is even) or $n = 2m + 1$ (if n is odd); furthermore, since $\delta = 2$, we set

$$\tilde{e} := \min\{i \geqslant 2 \mid 1 + q^2 + q^4 + \cdots + q^{2(e-1)} \equiv 0 \bmod \ell\}$$

in this case. We can now use the counting argument in the proof of [11, Theorem 4.11]; see also [9, §4]. This shows that the subsets $J \in \mathscr{P}_\sigma^*$ are parametrized (up to W-conjugacy) by the partitions $\lambda \vdash m$ such that each part of λ is equal to 1 or to $\tilde{e}\ell^j$ for some $j \geqslant 0$.

So the number of $J \in \mathscr{P}_\sigma^*$ (up to W-conjugacy) is equal to the number of sequences $(m_{-1}, m_0, m_1, \ldots)$ of non-negative integers such that $m_{-1} + \tilde{e}m_0 + \tilde{e}\ell m_1 + \cdots = m$. Thus, we find that the composition length of St_k for $G = \mathrm{GU}_n(q)$ is the coefficient of t^m (and not t^n as in Example 4.9) in the power series

$$\frac{1}{1-t}\prod_{j\geqslant 0}\frac{1}{1-t^{\tilde{e}\ell^j}} \qquad \text{(assuming that } \ell \text{ is linear for } G\text{).}$$

Remark 4.14. Within the much more general setting of Dipper–Gruber [6], we have considered here the "projective restriction system" $\mathscr{PR}(X_G, Y_L)$ where

$$X_G := \mathscr{S}_\sigma, \quad L = H \quad \text{and} \quad Y_L = \mathscr{O}H \;\; \text{(regular } \mathscr{O}H\text{-module).}$$

In this particular case, the arguments in [*loc. cit.*] drastically simplify, and this is what we have tried to present in this section. We note, however, that these methods only yield quite limited information about St_k when ℓ is not a "linear prime". Only two of the composition factors in the socle series displayed in Example 3.5 are accounted for by these methods (namely, k_G, φ_3 for ${}^2G_2(q^2)$ and k_G, ϑ for $\mathrm{GU}_3(q)$); all the remaining composition factors are cuspidal. Also note that, in these examples, St_k is not multiplicity-free.

Acknowledgements. I wish to thank Gerhard Hiss for clarifying discussions about the contents of [6], [16], [17].

References

[1] B. Ackermann, The Loewy series of the Steinberg-PIM of finite general linear groups, Proc. London Math. Soc. **92** (2006), 62–98.

[2] R. W. Carter, Finite groups of Lie type: conjugacy classes and complex characters, Wiley, New York, 1985; reprinted 1993 as Wiley Classics Library Edition.

[3] C. W. Curtis and I. Reiner, Methods of representation theory Vol. I, Wiley, New York, 1981.

[4] C. W. Curtis and I. Reiner, Methods of representation theory Vol. II, Wiley, New York, 1987.

[5] R. Dipper and J. Du, Harish-Chandra vertices, J. reine angew. Math. **437** (1993), 101–130.

[6] R. Dipper and J. Gruber, Generalized q-Schur algebras and modular representation theory of finite groups with split (B, N)-pairs, J. reine angew. Math. **511** (1999), 145–191.

[7] M. Geck, Modular Harish-Chandra series, Hecke algebras and (generalized) q-Schur algebras. *In*: Modular Representation Theory of Finite Groups (Charlottesville, VA, 1998; eds. M. J. Collins, B. J. Parshall and L. L. Scott), p. 1–66, Walter de Gruyter, Berlin 2001.

[8] M. Geck, Modular principal series representations, Int. Math. Res. Notices, vol. 2006, Article ID 41957, pp. 1–20.

[9] M. Geck and G. Hiss, Modular representations of finite groups of Lie type in non-defining characteristic. *In*: Finite reductive groups (Luminy, 1994; ed. M. Cabanes), Progress in Math. **141**, pp. 195–249, Birkhäuser, Boston , MA, 1997.

[10] M. Geck, G. Hiss and G. Malle, Cuspidal unipotent Brauer characters, J. Algebra **168** (1994), 182–220.

[11] M. Geck, G. Hiss and G. Malle, Towards a classification of the irreducible representations in non-defining characteristic of a finite group of Lie type, Math. Z. **221** (1996), 353–386.

[12] M. Geck and N. Jacon, Representations of Hecke algebras at roots of unity, Algebra and Applications **15**, Springer-Verlag, 2011.

[13] M. Geck and G. Pfeiffer, Characters of finite Coxeter groups and Iwahori–Hecke algebras, London Math. Soc. Monographs, New Series **21**, Oxford University Press, 2000.

[14] R. Gow, The Steinberg lattice of a finite Chevalley group and its modular reduction, J. London Math. Soc. **67** (2003), 593–608.

[15] J. A. Green, On a theorem of Sawada, J. London Math. Soc. **18** (1978), 247–252.

[16] J. Gruber, Cuspidale Untergruppen und Zerlegungszahlen klassischer Gruppen, Dissertation, Universität Heidelberg, 1995.

[17] J. Gruber and G. Hiss, Decomposition numbers of finite classical groups for linear primes, J. reine angew. Math. **485** (1997), 55–91.

[18] G. Hiss, The number of trivial composition factors of the Steinberg module, Arch. Math. **54** (1990), 247–251.

[19] G. Hiss, Harish-Chandra series of Brauer characters in a finite group with a split BN-pair, J. London Math. Soc. **48** (1993), 219–228.

[20] G. Hiss, Hermitian function fields, classical unitals, and representations of 3-dimensional unitary group, Indag. Math. **15** (2004), 223–243.

[21] R. B. Howlett and G. I. Lehrer, On Harish-Chandra induction for modules of Levi subgroups, J. Algebra **165** (1994), 172–183.

[22] N. Jacon and C. Lecouvey, On the Mullineux involution for Ariki–Koike algebras, J. Algebra **321** (2009), 2156–2170.

[23] G. D. James, Representations of general linear groups, London Math. Soc. Lecture Note Series, vol. 94, Cambridge University Press, Cambridge, 1984.

[24] G. D. James, The irreducible representations of the finite general linear groups, Proc. London Math. Soc. **52** (1986), 236–268.

[25] A. A. Khammash, A note on a theorem of Solomon–Tits, J. Algebra **130** (1990), 296–303.

[26] A. A. Khammash, On the homological construction of the Steinberg representation, J. Pure Appl. Algebra **87** (1993), 17–21.

[27] P. Landrock and G. O. Michler, Principal 2-blocks of the simple groups of Ree type, Trans. Amer. Math. Soc. **260** (1980), 83–111.

[28] A. Mathas, Iwahori-Hecke algebras and Schur algebras of the symmetric group, University Lecture Series, 15. Amer. Math. Soc., Providence, RI, 1999.

[29] T. Okuyama abd K. Waki, Decomposition numbers of $\mathrm{SU}(3, q^2)$, J. Algebra **255** (2002), 258–270.

[30] R. Steinberg, Prime power representations of finite linear groups II, Canad. J. Math. **9** (1957), 347–351.

[31] R. Steinberg, Lectures on Chevalley groups, Mimeographed Notes, Yale University, 1967; available at `http://www.math.ucla.edu/~rst/YaleNotes.pdf`

[32] F. Szechtman, Steinberg lattice of the general linear group and its modular reduction, J. Group Theory **14** (2011), 603–635.

[33] N. B. Tinberg, The Steinberg component of a finite group with a split (B, N)-pair, J. Algebra **104** (1986), 126–134.

IAZ—Lehrstuhl für Algebra, Universität Stuttgart, Paffenwaldring 57, 70569 Stuttgart, Germany

E-mail address: `meinolf.geck@mathematik.uni-stuttgart.de`